

HEMI-SLANT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

Mehraj Ahmad Lone^{a,*}, Mohamd Saleem Lone^b, Mohammad Hasan Shahid^c

^a*Department of Mathematics, Central University of Jammu, Jammu, 180011, India.*

^b*Department of Mathematics, Central University of Jammu, Jammu, 180011, India.*

^c*Department of Mathematics, Jamia Millia Islamia, New Delhi-110 025, India*

Abstract

In this paper we study the hemi-slant submanifolds of cosymplectic manifolds. Necessary and sufficient conditions for distributions to be integrable are worked out. Some important results are obtained in this direction.

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1. Introduction

In 1990, B. Y. Chen introduced the notion of slant submanifold, which generalizes holomorphic and totally real submanifolds [1]. After that many research articles have been published by different geometers in this direction for different ambient spaces.

A. Lotta introduced the notion of slant immersions of a Riemannian manifolds into an almost contact metric manifolds [3]. After, these submanifolds were studied by J. L. Cabrerizo *et. al* in the setting of Sasakian manifolds [7]. In [8] Papaghiuc defines the semi-slant submanifolds as a generalization of slant submanifolds. Bislant submanifolds of an almost Hermitian manifold were introduced as natural generalization of semi-slant submanifolds by Carriazo [2]. One of the classes of bi-slant submanifolds is that of anti-slant submanifolds which are studied by A. Carriazo [2] but the name anti-slant seems to refer that it has no slant factor, so B. Sahin [4] give the name of hemi-slant submanifolds instead of anti-slant submanifolds. In [2] V. A. Khan and M. A. Khan studied the hemi-slant submanifolds of sasakian manifolds.

In this paper we study the hemi-slant submanifolds of cosymplectic manifolds. In section 2, we collect the basic formulae and definitions for a cosymplectic manifolds and their submanifolds for ready references. In section 3, we study the hemi-slant submanifolds of cosymplectic manifolds. We obtain the integrability conditions of the distributions which are involved in the definition.

*Corresponding author

Email address: mehraj.jmi@gmail.com (Mehraj Ahmad Lone)

2. Preliminaries

Let N be a $(2m+1)$ -dimensional almost contact metric manifold with structure (ϕ, ξ, η, g) where ϕ is a tensor field of type $(1, 1)$, ξ a vector field, η is a one form and g is the Riemannian metric on N . Then they satisfy

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (1)$$

These conditions also imply that

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi), \quad (2)$$

and

$$g(\phi X, Y) + g(X, \phi Y) = 0, \quad (3)$$

for all vector fields X, Y in TN . Where TN denotes the Lie algebra of vector fields on N . A normal almost contact metric manifold is called a cosymplectic manifold if

$$(\bar{\nabla}_X \phi) = 0, \quad \bar{\nabla}_X \xi = 0, \quad (4)$$

where $\bar{\nabla}$ denotes the Levi-Civita connection of (N, g) .

Throughout, we denote by N a cosymplectic manifold, M a submanifold of N and ξ a structure vector field tangent to M . A and h denotes the shape operator and second fundamental form of immersion of M into N . If ∇ is the induced connection on M , the Gauss and Weingarten formulae of M into N are then given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (5)$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (6)$$

for all vector fields X, Y on TM and V on $T^\perp M$, where ∇^\perp denotes the connection on the normal bundle $T^\perp M$ of M . The shape operator and the second fundamental form are related by

$$g(A_V X, Y) = g(h(X, Y), V). \quad (7)$$

The mean curvature vector is defined by

$$H = \frac{1}{n} \text{trace}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i), \quad (8)$$

where n is the dimension of M and $\{e_1, e_2, \dots, e_n\}$ is the local orthonormal frame of M .

For any $X \in TM$, we can write

$$\phi X = TX + FX, \quad (9)$$

where TX and FX are the tangential and normal components of ϕX respectively.

Similarly for any $V \in T^\perp M$, we have

$$\phi V = tV + fV, \quad (10)$$

where tV and fV are the tangential and normal components of ϕV respectively. The covariant derivative of the tensor fields T, F, t and f are defined by the following

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (11)$$

$$(\nabla_X F)Y = \nabla_X^\perp FY - F\nabla_X Y, \quad (12)$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla_X^\perp V, \quad (13)$$

and

$$(\nabla_X f)V = \nabla_X^\perp fV - f\nabla_X^\perp V, \quad (14)$$

for all $X, Y \in TM$ and $V \in T^\perp M$.

A submanifold M of an almost contact metric manifold N is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H, \quad (15)$$

where H is the mean curvature vector. If $h(X, Y) = 0$ for any $X, Y \in TM$, then M is said to be totally geodesic and if $H = 0$, then M is said to be a minimal submanifold.

A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold [3] and slant submanifolds in Sasakian manifolds have been studied by J.L. Cabrerizo et al. [7].

For any $x \in M$ and $X \in T_x M$ if the vectors X and ξ are linearly independent, the angle denoted by $\theta(X) \in [0, \frac{\pi}{2}]$ between ϕX and $T_x M$ is well defined. If $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_x M$, we say that M is slant in N . The constant angle θ is then called the slant angle of M in N . The anti-invariant submanifold of an almost contact metric manifold is a slant submanifold with slant angle $\theta = \frac{\pi}{2}$ and an invariant submanifold is a slant submanifold with the slant angle $\theta = 0$. If the slant angle θ of M is different from 0 and $\frac{\pi}{2}$, then it is called a proper slant submanifold. If M is a slant submanifold of an almost contact manifold then the tangent bundle TM of M is decomposed as

$$TM = D \oplus \langle \xi \rangle,$$

where $\langle \xi \rangle$ denotes the distribution spanned by the structure vector field ξ and D is a complementary distribution of $\langle \xi \rangle$ in TM , known as the slant distribution. For a proper slant submanifold M of an almost contact manifold N with a slant angle θ , Lotta [3] proved that

$$T^2 X = -\cos^2 \theta (X - \eta(X)\xi), \quad \forall X \in TM.$$

Cabrerizo et al. [7] extended the above result into a characterization for a slant submanifold in a contact metric manifold. In fact, they obtained the following crucial theorems.

Theorem 2.1. [7] Let M be a slant submanifold of an almost contact metric manifold N such that $\xi \in TM$. Then M is slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that

$$T^2 = -\lambda(I - \eta \otimes \xi),$$

furthermore, in such case, if θ is the slant angle of M , then $\lambda = \cos^2 \theta$.

Theorem 2.2. [7] Let M be a slant submanifold of an almost contact metric manifold \overline{M} with slant angle θ . Then for any $X, Y \in TM$, we have

$$g(TX, TY) = \cos^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\},$$

and

$$g(FX, FY) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}.$$

3. Hemi-slant submanifolds of cosymplectic manifolds

In the present section, we introduce the hemi-slant submanifolds and obtain the necessary and sufficient conditions for the distributions of hemi-slant submanifolds of cosymplectic manifolds to be integrable.

Definition 3.1. Let M be submanifold of an almost contact metric manifold N , then M is said to be a hemi-slant submanifold if there exist two orthogonal distributions D^θ and D^\perp on M such that

- (i) $TM = D^\theta \oplus D^\perp \oplus \langle \xi \rangle$
- (ii) D^θ is a slant distribution with slant angle $\theta \neq \frac{\pi}{2}$,
- (iii) D^\perp is a totally real, that is $JD^\perp \subseteq T^\perp M$,

it is clear from above that CR-submanifolds and slant submanifolds are hemi-slant submanifolds with slant angle $\theta = \frac{\pi}{2}$ and $D^\theta = 0$, respectively.

In the rest of this paper, we use M a hemi-slant submanifold of almost contact metric manifold N .

On the other hand, if we denote the dimensions of the distributions D^\perp and D^θ by m_1 and m_2 respectively, then we have the following cases:

- (1) If $m_2 = 0$, then M is anti-invariant submanifold,
- (2) If $m_1 = 0$ and $\theta = 0$, then M is an invariant submanifold,
- (3) If $m_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold with slant angle θ ,
- (4) if $m_1, m_2 \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a proper hemi-slant submanifold.

Suppose M to be a hemi-slant submanifold of an almost contact metric manifold N , then for any $X \in TM$, we put

$$X = P_1X + P_2X + \eta(X)\xi, \quad (16)$$

where P_1 and P_2 are projection maps on the distribution D^\perp and D^θ . Now operating ϕ on both sides of (16), we arrive at

$$\phi X = \phi P_1X + \phi P_2X + \eta(X)\phi\xi,$$

Operating ϕ on both sides, we get

$$TX + FX = FP_1X + TP_2X + FP_2X,$$

It is easy to see on comparing that

$$TX = TP_2X, \quad FX = FP_1X + FP_2X,$$

If we denote the orthogonal complement of ϕTM in $T^\perp M$ by μ , then the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = F(D^\perp) \oplus F(D^\theta) \oplus \mu. \quad (17)$$

As $N(D^\perp)$ and $N(D^\theta)$ are orthogonal distributions on F . $g(Z, W) = 0$ for each $Z \in D^\perp$ and $W \in D^\theta$. Thus, by (1), (3) and (9), we have

$$g(FZ, FX) = g(\phi Z, \phi X) = g(Z, X) = 0, \quad (18)$$

which shows that the distributions $F(D^\perp)$ and $F(D^\theta)$ are mutually perpendicular. In fact, the decomposition (17) is an orthogonal direct decomposition.

Lemma 3.1. *Let M be a hemi-slant submanifolds of a cosymplectic manifold N . Then we have*

$$\nabla_X TY - A_{FY}X = T\nabla_X Y + th(X, Y)$$

and

$$h(X, TY) + \nabla_X^\perp FY = F\nabla_X Y + fh(X, Y)$$

for all $X, Y \in TM$.

Lemma 3.2. *Let M be a hemi-slant submanifolds of a cosymplectic manifold N . Then we have*

$$\nabla_X tV - A_{FV}X = -TA_VX + t\nabla_X^\perp V$$

and

$$h(X, tV) + \nabla_X^\perp FV = -fA_VY + f\nabla_V^\perp V.$$

for all $X \in TM$ and $V \in T^\perp M$.

Lemma 3.3. *Let M be a hemi-slant submanifolds of a cosymplectic manifold N , then*

$$h(X, \xi) = 0, \quad h(TX, \xi) = 0 \quad \nabla_X \xi = 0,$$

for all $X, Y \in TM$.

Proof. We know that for $\xi \in TM$, we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)$$

From (4), it follows that

$$\nabla_X \xi + h(X, \xi) = 0.$$

Thus result follows directly from the above equation. □

Theorem 3.4. *Let M be a hemi-slant submanifold of a cosymplectic manifold N , Then*

$$A_{\phi X} Y = A_{\phi Y} X,$$

for all $X, Y \in D^\theta$.

Proof. Using (7), we have

$$\begin{aligned} g(A_{\phi X} Y, Z) &= g(h(Y, Z), \phi X) \\ &= -g(\phi h(X, Z), X) \\ &= -g(\phi \bar{\nabla}_Z Y, X) - g(\phi \nabla_Z Y, X) \\ &= -g(\phi \bar{\nabla}_Z Y, X). \end{aligned}$$

Whereby using (4), we have

$$g(A_{\phi X} Y, Z) = -g(-A_{\phi Y} Z + \nabla_Z^\perp \phi Y, X).$$

By use of $h(X, Y) = h(Y, X)$, we arrive at

$$g(A_{\phi X} Y, Z) = g(A_{\phi Y} X, Z)$$

Hence the result. □

Theorem 3.5. *Let M be a submanifold of a cosymplectic manifold N . Then the distribution D^\perp is integrable if and only if*

$$A_{FZ} W = A_{FW} Z, \tag{19}$$

for any Z, W in D^\perp .

Proof. For $Z, W \in D^\perp$, by using (4), we have

$$(\bar{\nabla}_Z \phi)W = 0,$$

which implies that

$$\bar{\nabla}_Z \phi W - \phi \bar{\nabla}_Z W = 0.$$

Using (5), (6), (7) and (8), we have

$$\bar{\nabla}_Z FW - T\bar{\nabla}_Z W - F\bar{\nabla}_Z W = 0,$$

or

$$-A_{FW}Z - \nabla_Z^\perp FW - T\nabla_Z W + th(Z, W) - F\nabla_Z W - nh(Z, W) = 0, \quad (20)$$

Comparing the tangential components of (20), we have

$$A_{FW}Z + T\nabla_Z W + th(Z, W) = 0,$$

Interchange Z and W , and subtract, we have

$$T[Z, W] = A_{FW}Z - A_{FZ}W.$$

Thus $[Z, W] \in D^\perp$ if and only if (19) is satisfied □

Theorem 3.6. *Let M be a hemi-slant submanifold of a cosymplectic manifold N . Then the distribution $D^\theta \oplus D^\perp$ is integrable iff*

$$g([X, Y], \xi) = 0,$$

for all $X, Y \in D^\theta \oplus D^\perp$

Proof. For $X, Y \in D^\theta \oplus D^\perp$, we have

$$\begin{aligned} g([X, Y], \xi) &= g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \\ &= -g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \end{aligned}$$

Using (4), we have

$$g([X, Y], \xi) = 0$$

.

□

Theorem 3.7. *Let M be a hemi-slant submanifold of a cosymplectic manifold N . Then the anti-invariant distribution D^\perp is integrable if and only if*

$$T\nabla_Z W = T\nabla_W Z, \quad (21)$$

for any $X, Y \in D^\perp$.

Proof. For $X, Y \in D^\perp$, we have

$$(\nabla_Z \phi)W = 0,$$

or

$$\bar{\nabla}_Z \phi W - \phi \bar{\nabla}_Z W = 0,$$

whereby we have

$$\bar{\nabla}_Z FW - \phi(\nabla_Z W + h(W, Z)) = 0,$$

or

$$-A_{FW}Z + \bar{\nabla}_Z^\perp FW - T\nabla_Z W - F\nabla_Z W - th(Z, W) - nh(Z, W) = 0.$$

Comparing the tangential components we have,

$$-A_{FW}Z - T\nabla_Z W - th(Z, W) = 0,$$

from which we conclude that

$$T[Z, W] = A_{FW}Z + T\nabla_Z W + th(Z, W).$$

For $[Z, W] \in D^\perp$, we have $\phi[Z, W] = F[Z, W]$ because the tangential component of $\phi[Z, W]$ is zero. Thus, we have

$$A_{FW}Z + T\nabla_Z W + th(Z, W) = 0. \quad (22)$$

Similarly, we have

$$A_{FZ}W + T\nabla_W Z + th(W, Z) = 0. \quad (23)$$

Whereby use of Theorem 3, (22), and (23), we have

$$T\nabla_Z W = T\nabla_W Z$$

Thus the anti-invariant distribution D^\perp is integrable if and only if (21) is satisfied. \square

Theorem 3.8. *Let M be a hemi-slant submanifold of a cosymplectic manifold N . Then the slant distribution D^θ is integrable iff*

$$h(X, TY) - h(Y, TX) + \nabla_X^\perp FY - \nabla_Y^\perp FX \in \mu \oplus F(D^\theta),$$

for any $X, Y \in D^\theta$.

Proof. For $Z \in D^\perp$ and $X, Y \in D^\theta$, we have

$$g([X, Y], Z) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, Z).$$

Using (1), (2) and (4), we get

$$g([X, Y], Z) = g(\phi \bar{\nabla}_X Y, \phi Z) - g(\phi \bar{\nabla}_Y X, \phi Z)$$

whereby use of (5), (6), we obtain

$$g([X, Y], Z) = g(h(X, TY) - h(Y, TX) + \nabla_X^\perp FY - \nabla_Y^\perp FX, \phi Z)$$

As $\phi X \in \phi(D^\perp)$ and $F(D^\theta)$ and $F(D^\perp)$ are orthogonal to each other in $T^\perp M$, thus we conclude the result. \square

Theorem 3.9. *Let M be a hemi-slant submanifold of a cosymplectic manifold N . Then the slant distribution D^θ is integrable if and only if*

$$P_1 \{ \nabla_X TY - \nabla_Y TX - A_{FX} Y - A_{FY} X \} = 0,$$

for any $X, Y \in D^\theta$.

Proof. We denote by P_1 and P_2 the projections on D^\perp and D^θ respectively. For any vector fields $X, Y \in D^\theta$. Using equation (4), we have

$$(\bar{\nabla}_X \phi)Y = 0,$$

that is

$$(\bar{\nabla}_X \phi Y) - \phi \bar{\nabla}_X Y = 0.$$

By using equation (5), (6) and (9), we have

$$\bar{\nabla}_X TY + (\bar{\nabla}_X FY) - \phi(\nabla_X Y + h(X, Y)),$$

or

$$\nabla_X TY + h(X, TY) - A_{FY} X + \nabla_X^\perp FY - T \nabla_X Y - F \nabla_X Y - th(X, Y) - nh(X, Y) = 0.$$

Comparing the tangential components of (24), we have

$$\nabla_X TY - A_{FY} X - T \nabla_X Y - th(X, Y) = 0. \quad (24)$$

Replacing X and Y , we infer

$$\nabla_Y TX - A_{FX} Y - T \nabla_Y X - th(Y, X) = 0. \quad (25)$$

From (24) and (25), we arrive at

$$T[X, Y] = \nabla_X TY - \nabla_Y TX + A_{FY} X - A_{FX} Y. \quad (26)$$

Applying P_1 to (26), we obtain the result. \square

Theorem 3.10. *Let M be a hemi-slant submanifold of a cosymplectic manifold N . If the leaves of D^\perp are totally geodesic in M , then*

$$A_{FW}Z = \nabla_Z TW,$$

for $X \in D^\theta$ and $Z, W \in D^\perp$.

Proof. Since $(\bar{\nabla}_Z \phi)W = 0$. From (4), we have

$$\bar{\nabla}_Z \phi W = \phi \bar{\nabla}_Z W$$

Using (5), (6) and (9), we obtain

$$\nabla_Z TW + h(Z, TW) - A_{FW}Z + \nabla_Z^\perp FW = \phi \nabla_Z W + \phi h(Z, W).$$

For $X \in D^\theta$, we have

$$g(\nabla_Z TW, X) - g(A_{FW}Z, X) = g(\phi \nabla_Z W, X).$$

Therefore, we have

$$g(\nabla_Z W, \phi X) = g(A_{FW}Z - \nabla_Z TW, X). \quad (27)$$

The leaves of D^\perp are totally geodesic in M , if for $Z, W \in D^\perp$, $\nabla_Z W \in D^\perp$. Therefore from (27), we get the result. \square

Theorem 3.11. *Let M be a hemi-slant submanifold of a cosymplectic manifold N . If the leaves of D^θ are totally geodesic in M , then*

$$\nabla_X \phi Y = \phi h(X, Y),$$

for $X, Y \in D^\theta$ and $Z \in D^\perp$.

Proof. From (4), we know that $(\bar{\nabla}_X \phi)Y = 0$, then

$$\bar{\nabla}_X \phi Y = \phi \bar{\nabla}_X Y.$$

For $Z \in D^\perp$ and using (5), (6) and (9), we get

$$g(\nabla_X \phi Y, Z) - g(\phi \nabla_X Y, Z) = g(h(X, Y), \phi Z).$$

Therefore from above equation, we get the result. \square

Theorem 3.12. *Let M be a totally umbilical hemi-slant submanifold of a cosymplectic manifold N . Then at least one of the following holds*

- (1) $\dim(D^\perp) = 1$,
- (2) $H \in \mu$,
- (3) M is proper hemi-slant submanifold.

Proof. For a cosymplectic manifold, we have

$$(\bar{\nabla}_Z \phi)Z = 0,$$

for any $Z \in D^\perp$. Using (5),(6) and (9), we have

$$\bar{\nabla}_Z FZ - \phi(\nabla_Z Z + h(Z, Z)) = 0.$$

Whereby, we obtain

$$-A_{FZ}Z + \nabla_Z^\perp FZ - F\nabla_Z Z - th(Z, Z) - nh(Z, Z) = 0.$$

Comparing the tangential components, we have

$$A_{FZ}Z + th(Z, Z) = 0.$$

Taking inner product with $W \in D^\perp$, we obtain

$$g(A_{FZ}Z + th(Z, Z), W) = 0,$$

or

$$g(h(Z, W), FZ) + g(th(Z, Z), W) = 0.$$

Since M is totally umbilical submanifold, we obtain

$$g(Z, W)g(H, FZ) + g(Z, Z)g(tH, W) = 0.$$

The above equation has a solutio if either $\dim(D^\perp) = 1$ or $H \in \mu$ or $D^\perp = 0$, this completes the proof. \square

References

References

- [1] A. Carriazo, New developments in slant submanifold theory, *Narosa publisihing house, New delhi, India*, 2002.
- [2] A. Carriazo, Bi-slant immersions, *Proc. ICRAMS, Kharagpur, India*, 2000, 88-97.
- [3] A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Roumanie*, 1996, 39, 183-198.
- [4] B. Sahin, Warped product submanifolds of a Kaehler manifold with a slant factor, *Ann. Pol. Math.*, 2009, 95, 107-226.
- [5] B. Y. Chen, Geometry of Slant submanifolds, *Katholieke Universitiet Leuven*, 1990.
- [6] G. D. Ludden, Submanifolds of cosymplectic manifolds, *J. Differ. Geom.*, 4, 1970.
- [7] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Slant submanifolds in Sasakian manifolds, *Glasgow Math. J.*, 2000, 42, 125-138.
- [8] N. Papaghuic, Semi-slant submanifolds of a Kaehlarian manifold, *An. St. Univ. Al. I. Cuza. Iasi*, 2009, 40, 55-61.
- [9] R.S.Gupta, S. M. K. Haider, M. H.Shahid Slant submanifolds of Cosymplectic manifolds, *An. St. Univ. Al. I. Cuza. Iasi*, 2004, 50, 33-49.
- [10] K. Yano and M. Kon, Structures on Manifolds, *World Scientific, Singapore*, 1984.
- [11] V. A. Khan, M. A. Khan, Psuedo-slant submanifolds of a sasakian manifolds, *Indian J. Pure. and Applied Math.*, 2000, 38(1) 88-97.